General equilibrium in production

Suppose there are two firms: one produces good x and the other produces good y. Both firms use the factors of production K (capital) and L (labor), and face the prices p (output price), w (wage), and r (rental rate of capital). The production functions are

$$x = K_x^{1/2} L_x^{1/2}$$

$$y = K_y^{2/3} L_y^{1/3}$$

The total amount of capital is 3, and the total amount of labor is 2

1. Derive the production possibility frontier.
2. Find the equilibrium of the economy
3. Prove that the equilibrium is Pareto efficient

Solution

1. To derive the production possibility frontier, we maximize the output of good y for a given level of output x

   Let (K_x, L_x) be the inputs allocated to sector x, and (K_y, L_y) the inputs allocated to sector y

   Feasibility requires

$$K_x + K_y = 3$$

$$L_x + L_y = 2$$

   and output in sector x is fixed at

$$x = K_x^{1/2} L_x^{1/2}$$

   Hence, the planner's problem is

$$\max_{K_x, L_x} \ (3 - K_x)^{2/3} (2 - L_x)^{1/3}$$

   subject to

$$K_x^{1/2} L_x^{1/2} = x$$

   Since the logarithm is monotone, this is equivalent to maximizing

$$\frac{2}{3} \ln(3 - K_x) + \frac{1}{3} \ln(2 - L_x)$$

   subject to the same constraint

   The Lagrangian is

$$\mathcal{L} = \frac{2}{3} \ln(3 - K_x) + \frac{1}{3} \ln(2 - L_x) + \lambda \left(K_x^{1/2} L_x^{1/2} - x \right)$$

   The first-order conditions are

$$-\frac{2}{3(3 - K_x)} + \lambda \frac{1}{2} K_x^{-1/2} L_x^{1/2} = 0$$

$$-\frac{1}{3(2 - L_x)} + \lambda \frac{1}{2} K_x^{1/2} L_x^{-1/2} = 0$$

$$K_x^{1/2} L_x^{1/2} = x$$

   Dividing the first two conditions gives

$$\frac{2}{3 - K_x} \frac{K_x}{L_x} = \frac{1}{2 - L_x}$$

   which implies

$$2K_x(2 - L_x) = L_x(3 - K_x)$$

$$4K_x - 2K_xL_x = 3L_x - K_xL_x$$

$$4K_x - K_xL_x = 3L_x$$

$$4K_x = L_x(3 + K_x)$$

Thus, along the efficient allocation of factors,

$$L_x = \frac{4K_x}{3 + K_x}$$

Using this in the production function for good x,

$$x = \left(K_x \cdot \frac{4K_x}{3 + K_x}\right)^{1/2}$$

$$x(K_x) = \frac{2K_x}{\sqrt{3 + K_x}}$$

Now use the feasibility conditions to obtain the inputs used in sector y

$$K_y = 3 - K_x$$

$$L_y = 2 - \frac{4K_x}{3 + K_x}$$

$$L_y = \frac{2(3 - K_x)}{3 + K_x}$$

Therefore,

$$y = (3 - K_x)^{2/3}\left(\frac{2(3 - K_x)}{3 + K_x}\right)^{1/3}$$

$$y(K_x) = \left(\frac{2(3 - K_x)^3}{3 + K_x}\right)^{1/3}$$

Hence, the production possibility frontier can be written parametrically as

$$\begin{cases} x(K_x) = \dfrac{2K_x}{\sqrt{3 + K_x}} \\ y(K_x) = \left(\dfrac{2(3 - K_x)^3}{3 + K_x}\right)^{1/3} \end{cases} \qquad 0 \leq K_x \leq 3$$

The endpoints are obtained by assigning all factors to one sector
If $K_x = 0$, then

$$x = 0$$

$$y = \left(\frac{2 \cdot 3^3}{3}\right)^{1/3} = 18^{1/3} = 2^{1/3}3^{2/3}$$

If $K_x = 3$, then

$$x = \frac{2 \cdot 3}{\sqrt{6}} = \sqrt{6}$$

$$y = 0$$

So the frontier runs from

$$\left(0,\ 2^{1/3}3^{2/3}\right)$$

to

$$\left(\sqrt{6},\ 0\right)$$

Equivalently, solving for K_x as a function of x,

$$4K_x^2 - x^2 K_x - 3x^2 = 0$$

so

$$K_x(x) = \frac{x^2 + x\sqrt{x^2 + 48}}{8}$$

and therefore an explicit representation of the PPF is

$$y(x) = \left(\frac{2\left(3 - \frac{x^2 + x\sqrt{x^2+48}}{8}\right)^3}{3 + \frac{x^2 + x\sqrt{x^2+48}}{8}} \right)^{1/3}$$

with domain

$$0 \leq x \leq \sqrt{6}$$

Therefore, the production possibility frontier is the decreasing curve described by the parametric system above, connecting $\left(0, 2^{1/3}3^{2/3}\right)$ and $\left(\sqrt{6}, 0\right)$

2. Since the statement gives only the production side of the economy, but no preferences or final demand, the competitive equilibrium cannot pin down a unique relative output price. What we can do is characterize the *production equilibrium* as a function of the relative price of good x

   We normalize the price of good y to 1, and denote by p the price of good x

   Thus, firm x solves

$$\max_{K_x, L_x}\ pK_x^{1/2}L_x^{1/2} - rK_x - wL_x$$

   and firm y solves

$$\max_{K_y, L_y}\ K_y^{2/3}L_y^{1/3} - rK_y - wL_y$$

   with factor market clearing

$$K_x + K_y = 3 \qquad L_x + L_y = 2$$

Firm x

The first-order conditions are

$$\frac{p}{2}\sqrt{\frac{L_x}{K_x}} = r$$

$$\frac{p}{2}\sqrt{\frac{K_x}{L_x}} = w$$

Dividing the second equation by the first one, we obtain

$$\frac{K_x}{L_x} = \frac{w}{r}$$

Multiplying the two equations, we obtain

$$p = 2\sqrt{wr}$$

Firm y

The first-order conditions are

$$\frac{2}{3}K_y^{-1/3}L_y^{1/3} = r$$

$$\frac{1}{3}K_y^{2/3}L_y^{-2/3} = w$$

Dividing the second equation by the first one, we get

$$\frac{K_y}{L_y} = 2\frac{w}{r}$$

Since the technology has constant returns to scale, under perfect competition profits are zero, so price equals unit cost. For the y-sector this gives

$$1 = \frac{3}{2^{2/3}}r^{2/3}w^{1/3}$$

Let

$$\theta = \frac{w}{r}$$

Then the zero-profit condition of the y-sector becomes

$$1 = \frac{3}{2^{2/3}}r\theta^{1/3}$$

so

$$r = \frac{2^{2/3}}{3\theta^{1/3}}$$

Using now the zero-profit condition of the x-sector,

$$p = 2\sqrt{wr} = 2r\sqrt{\theta}$$

we get

$$p = 2\left(\frac{2^{2/3}}{3\theta^{1/3}}\right)\theta^{1/2}$$

$$p = \frac{2^{5/3}}{3}\theta^{1/6}$$

Hence,

$$\theta = \frac{729}{1024}p^6$$

Factor allocation

From the two firms' first-order conditions,

$$K_x = \theta L_x \qquad K_y = 2\theta L_y$$

Using factor market clearing,

$$\theta L_x + 2\theta L_y = 3$$

$$L_x + L_y = 2$$

Substituting $L_x = 2 - L_y$ into the first equation,

$$\theta(2 - L_y) + 2\theta L_y = 3$$

$$\theta(2 + L_y) = 3$$

$$L_y = \frac{3}{\theta} - 2$$

Therefore,

$$L_x = 2 - L_y = 4 - \frac{3}{\theta}$$

and then

$$K_x = \theta L_x = 4\theta - 3$$

$$K_y = 2\theta L_y = 6 - 4\theta$$

Thus, the competitive allocation of factors is

$$L_x^* = 4 - \frac{3}{\theta} \qquad K_x^* = 4\theta - 3$$

$$L_y^* = \frac{3}{\theta} - 2 \qquad K_y^* = 6 - 4\theta$$

with

$$\theta = \frac{729}{1024} p^6$$

Equilibrium output levels

The equilibrium quantities produced are

$$x^* = (K_x^*)^{1/2} (L_x^*)^{1/2}$$

$$x^* = (4\theta - 3)^{1/2} \left(4 - \frac{3}{\theta}\right)^{1/2}$$

$$x^* = \frac{4\theta - 3}{\sqrt{\theta}}$$

Similarly,

$$y^* = (K_y^*)^{2/3} (L_y^*)^{1/3}$$

$$y^* = (6 - 4\theta)^{2/3} \left(\frac{3}{\theta} - 2\right)^{1/3}$$

$$y^* = 2^{2/3} \frac{3 - 2\theta}{\theta^{1/3}}$$

Substituting $\theta = \frac{729}{1024} p^6$, we may also write the equilibrium output levels directly as functions of p

$$x^*(p) = \frac{243p^6 - 256}{72p^3}$$

$$y^*(p) = \frac{512 - 243p^6}{96p^2}$$

Factor prices

Using the formulas above,

$$r = \frac{2^{2/3}}{3\theta^{1/3}}$$

and

$$w = \theta r$$

so the equilibrium factor prices are

$$r^*(p) = \frac{16}{27p^2} \qquad w^*(p) = \frac{27p^4}{64}$$

For both sectors to produce nonnegative amounts, we need

$$K_x^* \geq 0 \qquad L_x^* \geq 0 \qquad K_y^* \geq 0 \qquad L_y^* \geq 0$$

which implies

$$\frac{3}{4} \leq \theta \leq \frac{3}{2}$$

equivalently,

$$\frac{2^{4/3}}{3^{5/6}} \leq p \leq \frac{2^{3/2}}{3^{5/6}}$$

Therefore, the economy has a continuum of competitive production equilibria, indexed by the relative price p. For each such p, the equilibrium factor prices, factor allocations, and outputs are given by the formulas above

3. Since this exercise only specifies the production side of the economy, the relevant notion of Pareto efficiency here is *production efficiency*

   That is, an equilibrium is Pareto efficient if the induced allocation of factors across the two sectors lies on the production contract curve, or equivalently, if the corresponding output pair lies on the production possibility frontier

   From part (1), the efficient allocations of factors satisfy

$$L_x = \frac{4K_x}{3 + K_x}$$

   Equivalently, the contract curve can be characterized by equality of the two firms' marginal rates of technical substitution

   For firm x,

$$x = K_x^{1/2} L_x^{1/2}$$

$$MPK_x = \frac{1}{2} K_x^{-1/2} L_x^{1/2} \qquad MPL_x = \frac{1}{2} K_x^{1/2} L_x^{-1/2}$$

   Hence,

$$MRTS_x = \frac{MPL_x}{MPK_x} = \frac{K_x}{L_x}$$

   For firm y,

$$y = K_y^{2/3} L_y^{1/3}$$

$$MPK_y = \frac{2}{3} K_y^{-1/3} L_y^{1/3} \qquad MPL_y = \frac{1}{3} K_y^{2/3} L_y^{-2/3}$$

   Hence,

$$MRTS_y = \frac{MPL_y}{MPK_y} = \frac{K_y}{2L_y}$$

   Therefore, production efficiency requires

$$\frac{K_x}{L_x} = \frac{K_y}{2L_y}$$

   Now, from part (2), the competitive equilibrium allocation satisfies

$$K_x^* = 4\theta - 3 \qquad L_x^* = 4 - \frac{3}{\theta}$$

$$K_y^* = 6 - 4\theta \qquad L_y^* = \frac{3}{\theta} - 2$$

with

$$\theta = \frac{w}{r}$$

Using the firms' first-order conditions obtained in part (2), we had

$$K_x^* = \theta L_x^*$$

$$K_y^* = 2\theta L_y^*$$

Therefore,

$$\frac{K_x^*}{L_x^*} = \theta$$

$$\frac{K_y^*}{2L_y^*} = \theta$$

and so

$$\frac{K_x^*}{L_x^*} = \frac{K_y^*}{2L_y^*}$$

Thus, the equilibrium allocation of factors satisfies the production-efficiency condition

Equivalently, substituting $K_x^* = 4\theta - 3$ into the contract-curve equation from part (1),

$$L_x = \frac{4K_x}{3 + K_x}$$

gives

$$L_x = \frac{4(4\theta - 3)}{3 + (4\theta - 3)}$$

$$L_x = \frac{4(4\theta - 3)}{4\theta}$$

$$L_x = 4 - \frac{3}{\theta}$$

which is exactly the equilibrium value L_x^*

So the competitive equilibrium lies on the contract curve, and therefore the associated output pair (x^*, y^*) lies on the production possibility frontier

Hence, there is no feasible reallocation of capital and labor across the two sectors that can increase the production of one good without reducing the production of the other

Therefore, the competitive equilibrium is Pareto efficient in production